INDIAN STATISTICAL INSTITUTE, BANGALORE

Back-paper Examination 2025–26 (First Semester)

M. Math. 1st Year **Algebra I**

Date: 24.12.2025 Marks: 100 Duration: 3 Hours

Instructions:

- Attempt all questions. Justify all your steps clearly. Feel free to use any result proved in class unless you have been asked to prove the same.
- R denotes a commutative ring with unity. All other notations are standard unless specified otherwise.

1. (a) Suppose that there exist ideals $I_1, \dots, I_n \subset R$ such that $\bigcap_{i=1}^n I_i = (0)$ and each R/I_i is a Noetherian ring. Prove that R is a Noetherian ring. [10+10]

   (b) Let $R \subset A \subset R[X]$ be commutative rings. Show that if A is Noetherian then so is R.

2. (a) Prove that $(M/IM) \otimes_{R/I} (N/IN) \cong (M \otimes_R N)/I(M \otimes_R N)$. [10+10]

   (b) Let A and B be commutative R-algebras. If A is a finitely generated R-algebra, then show that $A \otimes_R B$ is a finitely generated B-algebra. If both A and B are finitely generated over R, then prove that $A \otimes_R B$ is finitely generated as an R-algebra.

3. Let P be a projective R-module. [10+10]

   (a) If P is finitely generated R-module, then show that $\mathrm{Hom}_R(P, R)$ is a projective R-module.

   (b) Prove that there exists a free R-module F such that $P \oplus F$ is free.

4. Examine whether $\mathbb{Q}$ is (i) finitely generated as a $\mathbb{Z}$-module; (ii) decomposable as a $\mathbb{Z}$-module; (iii) projective as $\mathbb{Z}$-module. [5+5+10]

5. (a) Let R be a ring such that for every element $a \in R$, there exists an integer $n_a \geq 0$ such that $a^{n_a} = a$. Prove that every prime ideal of R is maximal. [10+10]

   (b) Prove that if I is an ideal of a Noetherian ring, then $(\sqrt{I})^n \subset I$ for some $n \geq 1$.